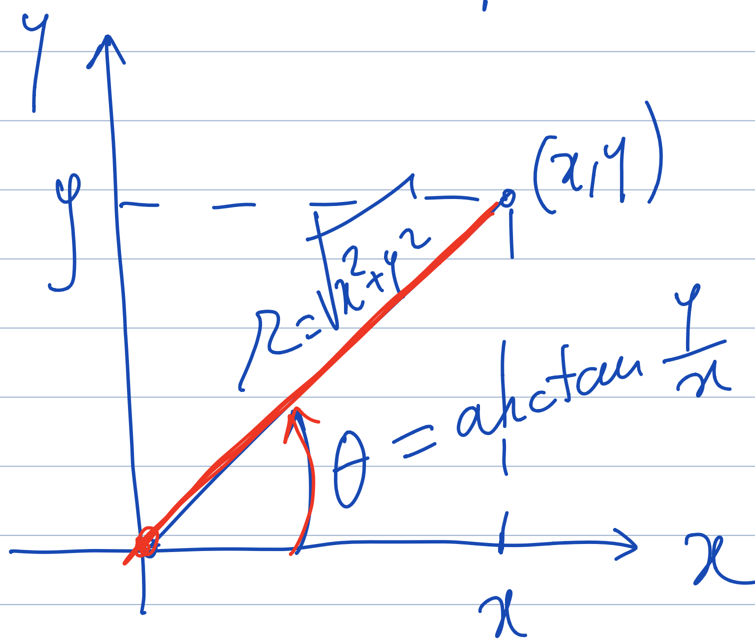
CDI-II – Prática-F7 20/4/21

Ficha 7. Mudança de Variáveis

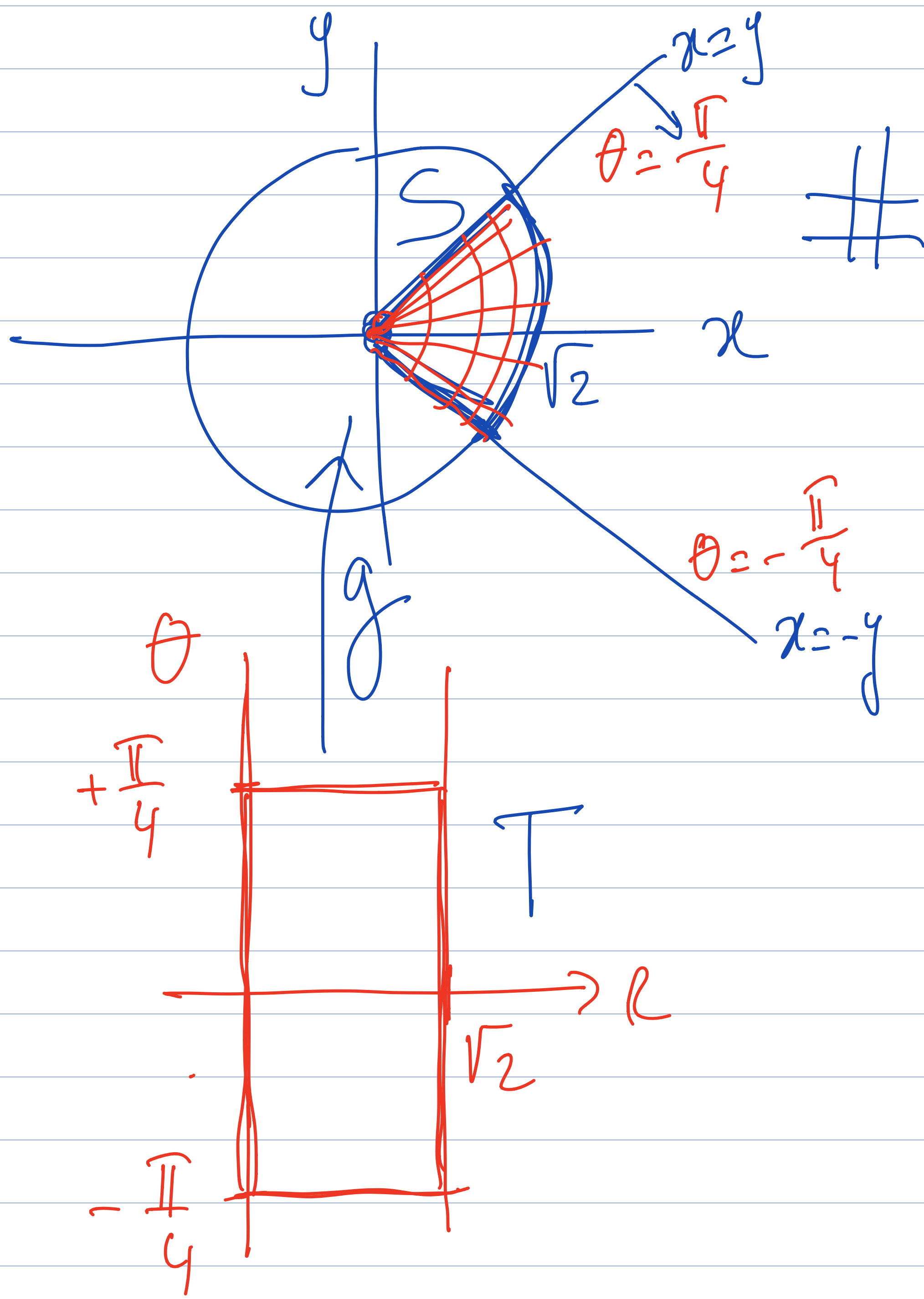
Coordenadas polares ($\mathbb{R}^2$)



$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

$$dx\,dy \longleftrightarrow r\,dr\,d\theta$$

1-a) $x^2+y^2<2$, $x>|y|$



$$\iint_S f(x,y)\,dx\,dy = \iint_T f(g(r,\theta))\,r\,dr\,d\theta$$

$$(x,y) = g(r,\theta) = (r\cos\theta, r\sin\theta)$$

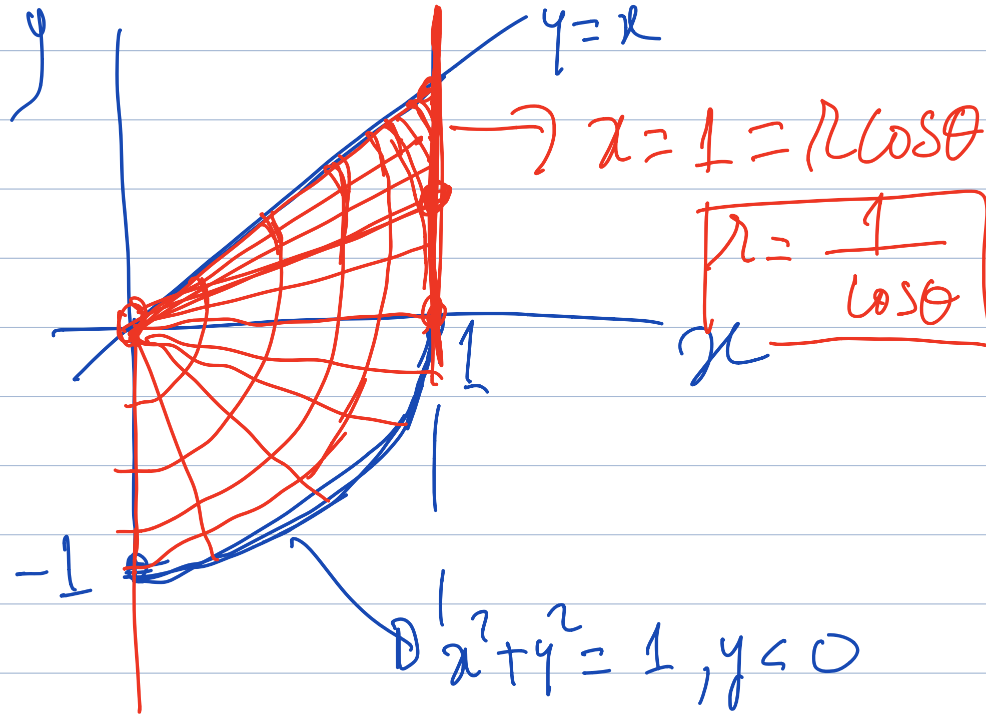
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\int_0^{\sqrt{2}} f(g(r,\theta))\,r\,dr \right) d\theta$$

II

1-c) $0 < x < 1$, $-\sqrt{1-x^2} < y < x$

$y = -\sqrt{1-x^2} \iff y^2 = 1-x^2;\ y<0$

$x^2+y^2=1,\ y<0$



$$\underbrace{\int_{-\frac{\pi}{2}}^{0}\left(\int_0^1 f(g(r,\theta))\,r\,dr\right)d\theta}_{\text{fácil}}+\underbrace{\int_0^{\frac{\pi}{4}}\left(\int_0^{\frac{1}{\cos\theta}} f(g(r,\theta))\,r\,dr\right)d\theta}_{\text{difícil!}}$$

$(x=1)$

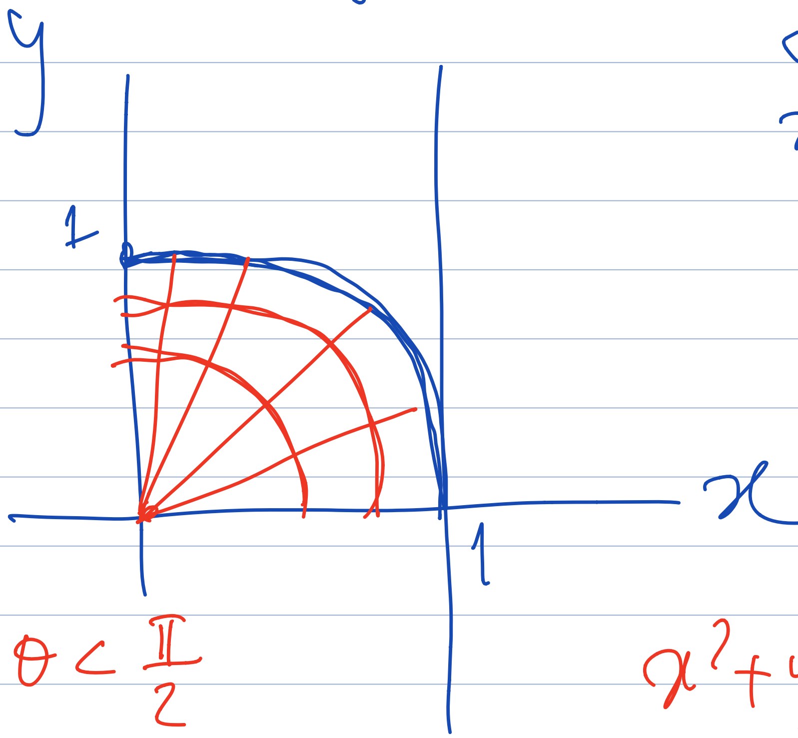
$$+\int_0^1\left(\int_0^x f(x,y)\,dy\right)dx$$

2-a) $0 < x < 1$

$0 < y < \sqrt{1-x^2}$

$y = \sqrt{1-x^2}$

$x^2+y^2=1,\ y>0$



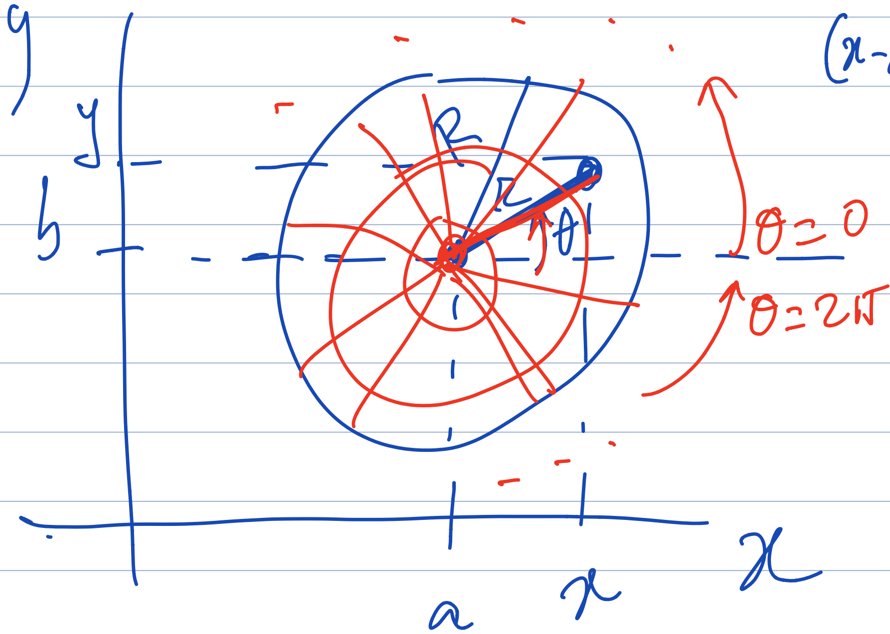
$$\begin{cases} 0 < \theta < \frac{\pi}{2} \\ 0 < r < 1 \end{cases}$$

$x^2+y^2=r^2$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\int_0^1 (-2r)e^{-r^2}dr\right)d\theta =$$

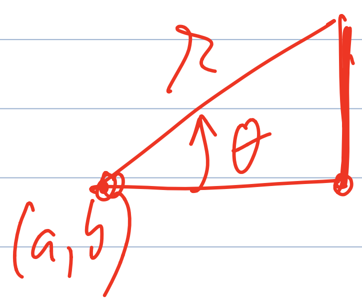
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(e^{-1}-1\right)d\theta = \frac{e^{-1}-1}{2}\frac{\pi}{2}$$

$(x-a)^2+(y-b)^2<R^2$



$$r^2 = (x-a)^2 + (y-b)^2 < R^2$$

$$\boxed{0 < r < R}$$



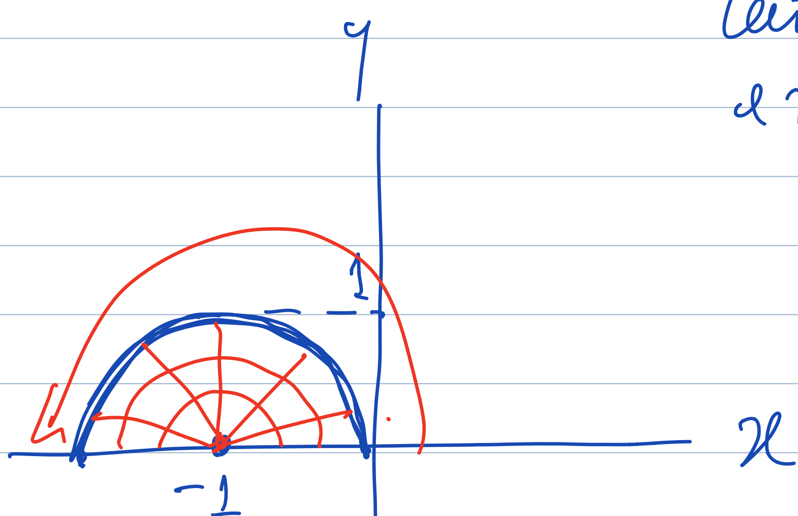
$$\begin{cases} x-a = r\cos\theta \\ y-b = r\operatorname{sen}\theta \end{cases} \quad \begin{cases} x = a + r\cos\theta \\ y = b + r\operatorname{sen}\theta \end{cases}$$

$$dx\,dy \longleftrightarrow r\,dr\,d\theta$$

2-c) $U: \underbrace{(\underbrace{x+1})^2 + y^2 < 1}_{} \;;\; y \geq 0$

Círculo de centro em $(-1,0)$ e raio 1.

$$\iint_U (x^2+y^2-1)\,dx\,dy$$



$$\begin{cases} 0 < r < 1 \\ 0 < \theta < \pi \end{cases}$$

r

$$\begin{cases} x+1 = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

$$\iint_{U} (x^2+y^2-1)\,dx\,dy =$$

$$= \int_0^{\pi}\left(\int_0^1\left((r\cos\theta-1)^2 + r^2 \operatorname{sen}^2\theta - 1\right) r\,dr\right)d\theta$$

$$= \int_0^{\pi}\left(\int_0^1\left(r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\operatorname{sen}^2\theta - 1\right) r\,dr\right)d\theta$$

$$= \int_0^{\pi}\left(\int_0^1\left(r^3 - 2r^2\cos\theta\right)dr\right)d\theta$$

etc...

2-d)

$$(x-1)^2+y^2<\frac{\pi^2}{4}$$

$$\begin{cases} x-1 = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

r

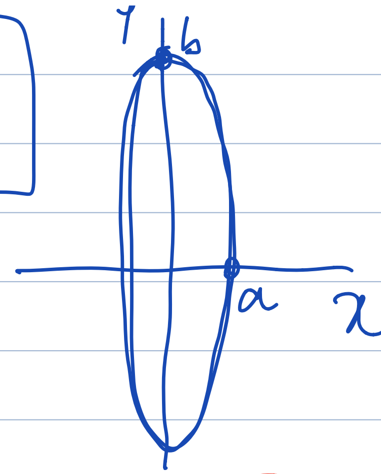
$$r^2<\frac{\pi^2}{4}$$

$$\begin{cases} 0<r<\frac{\pi}{2} \\ 0<\theta<2\pi \end{cases}$$

$$-\frac{1}{2}\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}(-2)r\,\text{sen}(r^2)\,dr\right)d\theta$$

etc . – –

$\boxed{\text{ELIPSE}}$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\le 1 \iff \boxed{r^2\le 1}$$

$$\begin{cases}\dfrac{x}{a}=r\cos\theta\\[2ex] \dfrac{y}{b}=r\,\mathrm{sen}\,\theta\end{cases}$$

$$\boxed{0<r<1}$$

$$\begin{cases} x=ar\cos\theta\\ y=br\,\mathrm{sen}\,\theta\end{cases}$$

$$(x,y)=g(r,\theta)=(ar\cos\theta,\ br\,\mathrm{sen}\,\theta)$$

$$\frac{y}{x} = \frac{b}{a}\tan\theta$$

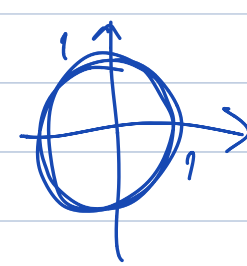
$$\frac{ay}{bx} = \tan\theta$$

$$\boxed{\theta = \arctan\left(\frac{ay}{bx}\right)}$$

$$\boxed{\det Dg(r,\theta) = abr}$$

<u>Nota:</u> $u = \frac{x}{a}$; $v = \frac{y}{b}$

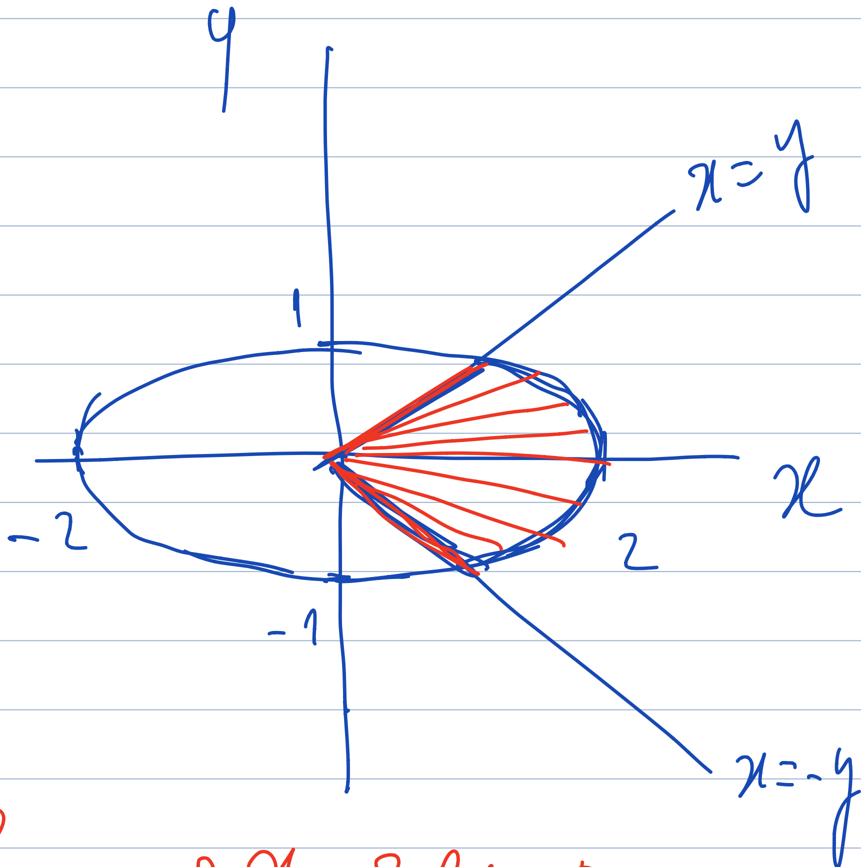
$$\Rightarrow \boxed{u^2 + v^2 < 1}$$



2-e) A: $\frac{x^2}{4} + y^2 \leq 1$

$x > |y|$

$vol_2(A) = ?$ (polares da elipse)



$a = 2$

$b = 1$

$$\begin{cases} x = 2r\cos\theta \\ y = r\sin\theta \end{cases}$$

$\boxed{2r}$

$$\theta = \arctan\left(\frac{2y}{x}\right)$$

$$x = y \longrightarrow \theta = \arctan(2)$$

$$x = -y \longrightarrow \theta = -\arctan(2).$$

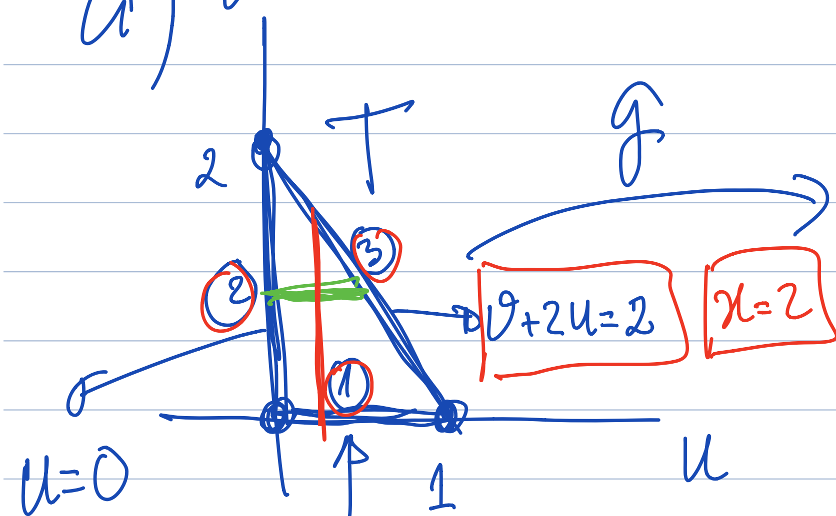
$$\mathrm{vol}_2(A) = \int_{-\arctan(2)}^{\arctan(2)} \left(\int_0^1 2r\,dr \right) d\theta$$

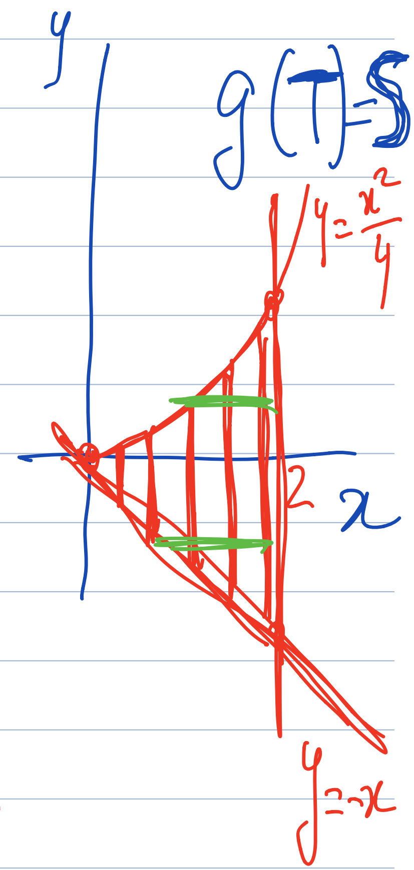
etc...

u

3 - $(x,y)=g(u,v)=(\boxed{2u+v},\ u^2-v)$

a)





$\boxed{v=0}$

$$\begin{cases} x = 2u \\ y = u^2 \end{cases}$$

$$\begin{cases} \frac{x}{2} = u \\ \boxed{y = \frac{x^2}{4}} \end{cases}$$

$u=0$

$$\begin{cases} x = v \\ y = -v \end{cases}$$

$$\boxed{y=-x}$$

3-b) $\displaystyle\iint \frac{1}{\sqrt{x+y+1}}\,dx\,dy =$

$f(x,y)$

$g(T) = S$

$$= \iint_T f\big(g(u,v)\big)\,\big|\det Dg(u,v)\big|\,du\,dv$$

$$\det Dg(u,v) = \det\begin{bmatrix} 2 & 1 \\ 2u & -1 \end{bmatrix} = -2-2u$$

$$\big|\det Dg(u,v)\big| = 2+2u$$

$$\frac{1}{\sqrt{x+y+1}} = \frac{1}{\sqrt{2u+\cancel{v}+u^2-\cancel{v}+1}}$$

$$= \frac{1}{\sqrt{(u+1)^2}} = \frac{1}{u+1}.$$

$$\iint_S f = \iint_T \frac{1}{u+1}(2+2u)\,du\,dv$$

$$= \iint_T 2\,du\,dv = 2\,\mathrm{vol}_2(T)$$

$$= 2 \times 1 = 2 \,//$$